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A Starting Algorithm for Minimal Cost Survivable Networks

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Contents

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Abstract: It is assumed that a network will be attacked by an all-knowing, intelligent enemy who will expend minimum effort to break communications in the network. An algorithm is presented which designs a near minimum cost network with specified incidences at each node.

INTRODUCTION

In designing a communication network which may be in danger of attack, there are three basic considerations: cost, the ability of the network to communicate after an attack, and the amount of the network structure that will be known by an attacker. If an attacker has no knowledge of the network, then any attack will be made at random and the network design should reflect this. In this report, however, we assume the attacker has complete knowledge of the network and attacks the most vulnerable points to disrupt communications. This assumption leads naturally to assuming that a good measure of survivability of a network is the fewest number of nodes which must be destroyed before the remaining nodes can no longer communicate with one another. Networks for which a great many nodes must be destroyed to minimally disrupt communications will be said to have high survivability.

One easy way to give a network high survivability is to insert many extra branches. However, this increases the overall cost of the network. The objective is a network which attains a specified level of survivability with minimal cost. At present the only known methods are algorithms for near minimal cost networks.

BACKGROUND

Suppose a network N is given. The maximum number of node-disjoint paths between a pair of nodes v_i and v_j is said to be the *redundancy* between v_i and v_j and is denoted r_{ij}. A set of nodes whose removal disconnects N (separates at least one node) is said to be an *articulation set.* The cardinality of the smallest articulation set of N (for N noncomplete) is said to be the *connectivity* of N and is denoted by ω. The smallest number of nodes whose removal disconnects all paths from nonadjacent nodes v_i and v_j is called the *i-j connectivity* and is denoted by ω_{ij}. Menger's theorem states in particular that ω_{ij} equals r_{ij}.

Since any articulation set is an i-j articulation set (a set of nodes which disconnects nodes v_i and v_j) for some pair of nodes v_i and v_j, it follows that

$$\omega = \min \omega_{ij},\ v_i,\ v_j \in N \text{ and } v_i,\ v_j \text{ nonadjacent.}$$

NRL Problem B01-10; Project RR 003-02-41-6153. This is a final report on one phase of the problem; work on this problem is continuing. Manuscript submitted March 8, 1971.

Hence

$$\omega = \min r_{ij},\ v_i,\ v_j \in N \text{ and } v_i,\ v_j \text{ nonadjacent.}$$

Since survivability is a measure of the minimal effort required to disconnect a network, we define the survivability of N to be ω, the connectivity. The survivability between pairs of nonadjacent nodes v_i and v_j is taken to be ω_{ij}, the i-j connectivity. (Equivalently it can be taken to be r_{ij}, the redundancy between v_i and v_j.)

Suppose a symmetric matrix $D = (d_{ij})$ is given with $d_{ii} = 0, i = 1, \ldots, n$, and each d_{ij} is a positive integer. Then any n-node network whose redundancies r_{ij} are such that $r_{ij} \geqslant d_{ij}$ $(i, j = 1, \ldots, n)$ is said to be *feasible.* (D is often called the redundancy matrix.) Suppose a cost matrix $C = (c_{ij})$ is given, where c_{ij} is the cost of a branch between nodes v_i and v_j. C is assumed to be symmetric with all zeros on the main diagonal.

The problem is to find a feasible network of minimal cost, called a *solution network* (or an *optimal network).* In general no optimal network is unique. Indeed, when C is a constant matrix, the problem reduces to finding a feasible network with the fewest branches.

There is an algorithm due to Steiglitz, Weiner, and Kleitman [1] which can be used to find a near optimal network. This algorithm has two parts. First is a starting routine which constructs a feasible network. Second is an optimizing routine which searches networks generated by a pair-branch local transformation for a feasible network of lower cost. When a locally transformed network of lower cost and feasibility is found, the improved network is adopted. The search continues with the new networks by local transformations until no further cost improvements can be found.

The aim of this report is to present an improved starting routine with the advantages of flexibility in the number of branches chosen for the network and of lower cost.

THE STARTING ROUTINE

Suppose that n distinct points are given, along with an n by n redundancy matrix $R = (r_{ij})$ and an n by n cost matrix $C = (c_{ij})$. Let $R_i = \max_j r_{ij}$ be the *row incidence.* Let C be considered as the sum of two triangular matrices U and L, where $c_{ij} \in U$ if $i < j$ and $c_{ij} \in L$ if $i > j$:

$$C = \begin{pmatrix} 0 & & U \\ & \ddots & \\ L & & 0 \end{pmatrix}.$$

This can be done, since C is symmetric with a zero main diagonal. The aim is to construct a simple low-cost network so that the incidence at the node v_i is R_i. This latter condition is necessary, since the degree of the node v_i must be at least $\max_{i,j} r_{ij}$ if at least r_{ij} distinct paths are to be demanded between v_i and v_j. This condition, however, is not sufficient for feasibility. Thus each network constructed must be tested by the method of Frisch [2], which calculates the node-pair redundancy. For this algorithm it is assumed that the R_i values are known.

Consider all the possible sums of length R_1 of elements of the first row of matrix C. In general many of these sums will have the minimal sum value. Let S_1 (indicating stage 1 in the routine) be the collection of elements which appear in any of these minimal sums. S_1 in general will look like

$$c_{1t_1} \leqslant c_{1t_2} \leqslant \ldots \leqslant c_{1t_k} < c_{1\ell_1} = c_{1\ell_2} = \ldots = c_{1\ell_{R_1''}}.$$

The elements from ℓ_1 to $\ell_{R_1''}$ are called optimal elements. (In the example given in Appendix A, $S_1 = \{2, 1, 1\}$.) R_1'' is the number of optional elements.

The strategy is to select R_1 elements from S_1 to go "into" the network, i.e., into class J. (For example, if c_{13} is chosen, then c_{13} is put in J and branch (v_1, v_3) is put into the network.) First place all the nonoptional elements of S_1 into J. The problem is to decide which optional elements to put into J. (The first row is the easiest, since there are no tradeoffs.)

Consider the elements

$$c_{\ell_1 1},\ c_{\ell_2 1},\ \ldots,\ c_{\ell_m 1} \qquad (m = R_2'').$$

These are the reflections of the optional elements. In the rows $\ell_1, \ldots \ell_m$ there are sets

$$S_{\ell_1}, \ldots, S_{\ell_m}$$

and optional element sets $O_{\ell_1}, \ldots, O_{\ell_m}$.

If any of the $c_{\ell_i 1}$ belong to $S_{\ell_i} \setminus O_{\ell_i}$, then note (choose) this c_{ℓ_1} and put the corresponding $c_{1\ell_i}$ into J. To make sure no more than R_1 elements are placed into J, note the $c_{\ell_i 1}$ in $S_{\ell_i} \setminus O_{\ell_i}$ by decreasing order of the corresponding R_{ℓ_i}. Thus the $c_{\ell_i 1}$ is noted first whose R_{ℓ_i} is greatest. (In case of ties choose that $c_{\ell_i 1}$ whose row sum is largest.) (In the example $R_1 = 3$ and the optional set $O_1 = \{2\}$. Since $|S_1| = 3 = R_1$, all of S_1 is placed into J.)

On the other hand, suppose that the number of $c_{\ell_i 1}$ which belongs to $S_{\ell_i} \setminus O_{\ell_i}$ is insufficient, so that less than R_1 elements are placed into J. In this case pick the $c_{1\ell_i}$ so that the $c_{1\ell_i}$ belong to rows with the largest R_{ℓ_i} value. In case of a tie note those $c_{\ell_i 1}$ from rows with the largest optional sets. If a tie persists, choose at random. (This last situation does not arise in the example.)

Let K be the set of elements symmetric to those selected for J. Elements in K will be called *blue* elements, and elements in J will be called *orange* elements. (An element in K or in J will be called *circled.*)

Now proceed to the second row of the C matrix. Some elements in this row may be blue elements already. Thus let $R_2' = R_2$ minus the number of blue elements. In general $R_j' = R_j$ minus the number of blue elements, in the jth row.

Consider all possible sums of length R_2' of uncircled elements of row two. Let M_2 be the collection of all those sums of minimal value, and let S_2 be the set of all elements which appear in some sum belonging to M_2. In general S_2 will consist of the smallest uncircled elements plus a number of optional elements which are all equal (in magnitude).

Let

$$O_2 = \{c_{2\ell_1},\ c_{2\ell_2}, \ldots,\ c_{2\ell_m}\}$$

be the optional elements, where $m = R_2''$, the number of optional elements. (In the example $R_2 = 3$, $R_2' = 2$, the element $c_{21} = 1$ is blue, $S_2 = \{1, 2, 2\}$, $O_2 = \{2, 2\}$.)

The problem is to determine which of the optional elements to put in J. Consider the elements symmetric to the optional elements:

$$c_{\ell_1 2},\ c_{\ell_2 2},\ \ldots,\ c_{\ell_2 m} \qquad (m = R_2'').$$

Let S_{ℓ_i} $(i = 1, \ldots, R_2'')$ be the set of uncircled elements in the ℓ_ith row $(i = 1, \ldots, R_2'')$ which belong to minimal sums of length R_{ℓ_i}'. (In general, S_{ℓ_i} contains more than R_{ℓ_i}' elements due to the optional elements.)

Define *Case I* to be the case in which there are at least R_2' elements in $S_2 \cap U$, where U is the upper triangular submatrix of C. Let b_j be the number of elements in the basic set $B_j = S_j \setminus O_j$. Let R_2''' be the number of optional elements in O_2 needed for J in order for the second row to have R_2 circled elements, (considering B_2 as already in J). Let N_2 be the number of elements in O_2 for which $c_{\ell_i 2} \in B_{\ell_i}$ (the basic set). If $R_2' = 0$, go to the next stage by changing the 2's to 3's in the algorithm.

There are two subcases in Case I:

A. If $N_2 \geqslant R_2'''$, then note the $c_{\ell_j 2}$ in B_{ℓ_j} in order of decreasing R_{ℓ_j}' values. Put in J those $c_{2\ell_j}$ whose reflections $c_{\ell_j 2}$ have been noted above. In case of a tie avoid noting those $c_{\ell_j 2}$ which belong to rows whose row sums are largest.

B. If $N_2 < R_2'''$, then let N_2' be the number of elements in O_2 for which $c_{\ell_i 2} \in S_{\ell_i}$. There are two subcases to be considered in this subcase:

1. If $N_2' \geqslant R_2'''$, then place in J all elements in B_2 and the R_2''' elements in O_2 for which the corresponding O_{ℓ_i} are largest (this takes care of the case when $S_{\ell_i} = \Phi$) making sure that if $c_{2\ell_j}$ is in O_2 then $c_{2\ell_j}$ is placed in J whenever $c_{\ell_j 2}$ is in B_{ℓ_j} and moreover, that a $c_{2\ell_j}$ in O_2 is placed in J only if $c_{\ell_j 2}$ is in S_{ℓ_j}. In case of a tie avoid noting the $c_{\ell_j 2}$ in rows with the largest row sum. If a tie persists choose at random. (In the example $N_2 = O, R_2''' = 1, N_2' = 1, O_6 = \{2, 2, 2\}$, and $O_2 = \Phi$. Hence c_{62} is noted, c_{26} is placed into J, and c_{62} is placed into K.)

2. $N_2 < R_2'''$, then place into J those R_2''' elements $c_{2\ell_i}$ belonging to O_2 whose reflections $c_{\ell_i 2}$ increase the cost of the ℓ_ith row the least (over the rows theoretical minimum) making sure that if $c_{2\ell_j}$ is in O_2 then $c_{2\ell_j}$ is placed in J whenever $c_{\ell_j 2}$ is in S_{ℓ_j}. Also, place all the elements in B_2 into J. Add to K the reflections of these new orange elements.

Define *Case II* to be the case in which there are not R_2' uncircled elements in $S_2 \cap U$. In this case check each element in $L \cap S_2$ for a tradeoff advantage. Consider the element c_{2k} in L. A tradeoff advantage occurs when the difference between the cost of some uncircled elements in $U \cap S_2$ and $L \cap S_2$, say $c_{2j} - c_{2k}$ ($j > 2, k < 2$, so $c_{2k} \in L, c_{2j} \in U$), is less than or equal to the largest element in S_j.

If this occurs, choose the c_{2j} for J which gave rise to the greatest tradeoff difference. (If $c_{j2} \in B_j$ and $c_{2j} - c_{2k} \geqslant O$, where c_{2k} is the smallest element in $S_2 \cap L$, then always choose c_{2j} for J.) Place the corresponding reflected element c_{j2} into K. Lower R_2' by one, and recycle through the algorithm with this new R_2' values. Make sure that in the tradeoff the number of circled elements in the jth row does not exceed R_j. If tradeoff does not occur, then choose all elements in $S_2 \cap U$ for J and choose the smallest in $S_2 \cap L$ for J until R_2' elements have been circled.

Proceed to the third row, changing the appropriate 2's to 3's. Repeat this one stage at a time until the nth row has been processed.

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Appendix A

EXAMPLE

In the following example the R_i are assumed to be given as follows: $R_1 = 3, R_2 = 3, R_3 = 1, R_4 = 3, R_5 = 3, R_6 = 4, R_7 = 3$ and $R_8 = 4$.

STAGE 1

In stage 1 of the algorithm as applied to the example of the C matrix to follow it is given that $R_1 = 3$ and, since the first row has no blue elements, $R_1' = 3$. Hence in the example of the matrix to follow $S_1 = \{2, 1, 1\}$ and $O_1 = \{2\}$. Thus place all of S_1 into J (and their reflections into K). (A heavy circle about an element indicates that the corresponding edge belongs to the orange set (i.e., J set). A light circle indicates an edge which belongs to the blue set.)

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 2

For the second row of the above matrix $R_2 = 3$ and $R_2' = 2$. Hence $S_2 = \{2, 2, 1\}$. Since $O_2 = \{2, 2\}$, the problem is whether c_{23} or c_{26} should be placed in J. Consider the elements symmetrical to these: $c_{32} \in S_3 \setminus O_3$ and $c_{62} \in S_6 \setminus O_6$. Under subcase B of Case I, choose those $c_{2\ell_i}$ corresponding to the largest O_{ℓ_i}: $O_6 = \{2, 2, 2\}$ and $O_3 = \Phi$. Thus c_{26} is placed in J, and the matrix now has circled elements as follows.

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 3

At this stage $R_3 = 1, R'_3 = 0$, and the matrix remains the same as at the end of stage 2; thus proceed to stage 4.

STAGE 4

For the fourth row of the matrix in the preceding diagram, $R'_4 = 2$. Hence $S_4 = \{1, 1, 1\}$ and $O_4 = \{1, 1, 1\}$. Since $B_4 = \Phi$ and $S_4 \cap U = \{1, 1\}$, then c_{45} and c_{48} are placed in J (and c_{54} and c_{84} are placed in K). The matrix becomes as follows.

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 5

For the fifth row of the matrix $R_5 = 3$ and $R'_5 = 2$. Hence $S_5 = \{1, 1, 1\} = \{c_{56}, c_{57}, c_{58}\}$ and $O_5 = \{1, 1, 1\}$. In choosing two of the three optional elements note that $B_5 = \Phi, N_2 = 3, R_2''' = 2$, and c_{65} and c_{85} belong to the rows with the highest row incidence. Thus c_{56} and c_{58} are placed in J (and c_{65} and c_{85} are placed into K), so that the matrix becomes as follows.

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 6

For the sixth row of the matrix $R_6 = 4$ and $R'_6 = 2$. Hence $S_6 = \{2, 2, 2\}$. But $S_6 \cap U = \{2, 2\} = \{c_{67}, c_{68}\}$. Thus c_{67} and c_{68} are placed in J (and c_{76} and c_{86} are placed in K), so that the matrix becomes as follows.

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 7

For the seventh row of the matrix $R_7 = 3$ and $R'_7 = 1$. Hence $S_7 = \{1\}$ and $O_7 = \Phi$. In this case, $S_7 \cap U$ does not have R'_7 elements. Thus, there is the possibility of a tradeoff: $c_{78} - c_{75} = 3 - 1 = 2$. However, 2 is not less than the largest element in S_8, namely 1. Hence no tradeoff advantage is available; c_{75} is placed into J (and c_{57} is placed in K), so that the matrix becomes as follows.

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

STAGE 8

In the eighth row of the matrix $R_8 = 4$ and $R'_8 = 1$. Hence $S_8 = \{1\} = \{c_{83}\}$. Since $S_8 \cap U = \Phi$, there is no tradeoff possibility. Thus, place c_{83} in J (and c_{38} in K), so that the matrix becomes as shown below. (Placing c_{38} in K gives two circled elements in row 3 and exceeds the row incidence by one.)

	1	2	3	4	5	6	7	8
1		1	1	2	4	4	8	3
2	1		2	6	3	2	1	4
3	1	2		1	5	3	2	1
4	2	6	1		1	2	2	1
5	4	3	5	1		1	1	1
6	4	2	3	2	1		2	2
7	8	1	2	2	1	2		3
8	3	4	1	1	1	2	3	

The total cost equals 1 + 1 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 1 + 1 = 17.

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